28.(a). Derive (6.19) by using the usual coordinate transformation from Cartesian to spherical polars.

(6.19): $g_{ij} = \begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$ in polars.

$g_{\alpha'\beta'} = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'}$

$= \Lambda^i_{\alpha'} \vec{e}_i \cdot \Lambda^j_{\beta'} \vec{e}_j$

$\Rightarrow$ ~~$g_{rr} =$~~ $x^1 = r\sin\theta\cos\phi$

$x^2 = r\sin\theta\sin\phi$

$x^3 = r\cos\theta$

$\Rightarrow \dfrac{dx^1}{dr} = \sin\theta\cos\phi$

$\dfrac{dx^2}{dr} = \sin\theta\sin\phi$

$\dfrac{dx^3}{dr} = \cos\theta$

$\Rightarrow g_{rr} = \sin^2\theta\cos^2\phi\ \cancel{\vec{e}_1\cdot\vec{e}_1}\ \delta_{11}$

$+ \sin^2\theta\sin^2\phi\ \cancel{\vec{e}_2\cdot\vec{e}_2}\ \delta_{22}$

$+ \cos^2\theta\ \cancel{\vec{e}_3\cdot\vec{e}_3}\ \delta_{33}$

$= \boxed{1}$

$\dfrac{dx^1}{d\theta} = r\cos\theta\cos\phi$

$\dfrac{dx^2}{d\theta} = r\cos\theta\sin\phi$

$\dfrac{dx^3}{d\theta} = -r\sin\theta$

$\Rightarrow g_{\theta\theta} = r^2\cos^2\theta\cos^2\phi\ \delta_{11}$

$+ r^2\cos^2\theta\sin^2\phi\ \delta_{22}$

$+ r^2\sin^2\theta\ \delta_{33}$

$= \boxed{r^2}$

$\dfrac{dx^1}{d\phi} = -r\sin\theta\sin\phi$

$\dfrac{dx^2}{d\phi} = r\sin\theta\cos\phi$

$\dfrac{dx^3}{d\phi} = 0.$

$g_{\phi\phi} = r^2\sin^2\theta\sin^2\phi\ \delta_{11}$

$+ r^2\sin^2\theta\cos^2\phi\ \delta_{22}$

$+ 0$

$= \boxed{r^2\sin^2\theta}$

$\Rightarrow g_{ij}$ is $\begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$ in spherical.

(b) Deduce from (6.14) that the metric of the surface of a sphere of radius $r$ has components ($g_{\theta\theta} = r^2$, $g_{\phi\phi} = r^2 \sin^2\phi$, $g_{\theta\phi} = 0$) in the usual spherical coordinates.

A sphere of radius is a 2-manifold parameterized by $\theta, \phi$. Let it is also a special case described by spherical coordinates $\begin{pmatrix} r \\ \theta \\ \phi \end{pmatrix}$ with fixed $r$.

$\Rightarrow$ we take the metric of spherical coordinates and remove redundancy in consideration of $\vec{e}_r$ and $\tilde{\omega}_r$, that is, basis vector and 1-form for $r$. and obtain

$$\boxed{g = \begin{pmatrix} r^2 & \\ & r^2 \sin^2\theta \end{pmatrix}}$$

(c) Find the components $g^{\alpha\beta}$ for the sphere:

As above,

$$g^{\alpha\beta} = \left(g_{\alpha\beta}\right)^{-1}$$

$$= \begin{pmatrix} r^2 & \\ & r^2 \sin^2\phi \end{pmatrix}^{-1}$$

$$\boxed{= \begin{pmatrix} r^{-2} & \\ & r^{-2} \sin^{-2}\theta \end{pmatrix}}$$